STAT0041: Stochastic Calculus

Lecture 7 - Properties of Brownian Motion

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Key concepts:

- *Martingale properties;*
- *Sample path's properties.*

7.1 Martingale properties

Consider

$$\mathscr{F}_t := \sigma(\{B_s : 0 \le s \le t\}).$$

is natural filtration of $(B_t)_{t\ge0}$

Proposition 7.1 *Let (B_t) be one-dimension standard Brownian motion, then he following are $\mathscr{F}_t$-martingales:*

(1) B_t;

(2) $B_t^2 - t$;

(3) $\exp(\lambda B_t - \frac{1}{2}\lambda^2 t)$, $\forall\ \lambda \in \mathbb{R}$.

$\exp(\lambda B_t - \frac{1}{2}\lambda^2 t)$ is known as *exponential martingale* of Brownian motion.

7.2 Sample path's properties

We have defined Brownian motion as a stochastic process $(B_t)_{t\ge0}$ which is just a family of random variables $\omega \mapsto B(t,\omega)$ defined on a single probability space. At the same time, a stochastic process can also be interpreted as a random function with the sample functions defined by $t \mapsto B(t,\omega)$. The *sample path properties* of a stochastic process are the properties of these random functions.

Theorem 7.2 *Almost surely, Brownian motion is nowhere differentiable.*

Definition 7.3 (Variation) *p-variation of a continuous function $f : [0,t] \to \mathbb{R}$ is*

$$V_f^{(p)}(t) := \sup \sum_{j=1}^{k} |f(t_j) - f(t_{j-1})|^p ,$$

where the supremum is over all $k \in \mathbb{N}$ and partitions $0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_{k-1} \leqslant t_k = t$.

Corollary 7.4 *Since Brownian motion is almost surely non-differentiable and continuous, it is of unbounded variation.*[1]

Although Brownian motion is of unbounded variation, its quadratic variation is t in the sense of L^2, which is the key to defining stochastic integral.

Theorem 7.5 *Let (B_t) be one-dimension standard Brownian motion, and*

$$0 = t_0^{(n)} \leqslant t_1^{(n)} \leqslant \cdots \leqslant t_{k(n)-1}^{(n)} \leqslant t_{k(n)}^{(n)} = t$$

be partition of $[0,t]$, and $\Delta(n) := \sup_{1\leqslant j\leqslant k(n)} \left\{ t_j^{(n)} - t_{j-1}^{(n)} \right\}$, then as $\Delta(n) \to 0$,

$$\mathbb{E} \left| \sum_{j=1}^{k(n)} \left(B_{t_j^{(n)}} - B_{t_{j-1}^{(n)}} \right)^2 - t \right|^2 \to 0.$$

We can conclude same result for almost surely convergence (Theorem 1.35 in [1]). The proof relies on *Hölder continuous* property of Brownian motion, that is: there almost surely exist $c(\omega)$, such that

$$|B_t(\omega) - B_s(\omega)| \leq c(\omega)|t-s|^{\alpha}, \quad \forall\, \alpha \in [0, \frac{1}{2}).$$

References

[1] Mörters Peter, and Yuval Peres. Brownian motion. Vol. 30. Cambridge University Press, 2010.

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[1] We have the following chains of inclusions for continuous functions over a closed, bounded interval of the real line:

Continuously differentiable $\subset$ Lipschitz continuous $\subset$ absolutely continuous $\subset$ continuous and bounded variation $\subset$ differentiable almost everywhere